One-parameter family of equations of state for isotropic compressible solids

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Abstract

Applying the theorem proved by the authors in [10], we established the hyperbolicity of non-stationary equations of hyperelastic isotropic solids for a one-parameter family of equations of state containing, in particular, generalized neo-hookean solids. The hyperbolicity is equivalent to the rank-one convexity of the corresponding stored energy. The influence of the parameter on the solution properties is shown in the case of a strong shear test.

Key words : hyperelasticity, hyperbolicity, neo-hookean materials

1 Introduction

Hyperelastic compressible solids are characterized by a specific stored energy e. We take the energy in separable form [7] :

$$e\left(\mathbf{G},\eta\right)=e^{h}\left(\rho,\eta\right)+e^{e}\left(\mathbf{g}\right). \tag{1}$$

Here $\mathbf{G}=\mathbf{B}^{-1}$ is the Finger tensor, $\mathbf{B}=\mathbf{F}\mathbf{F}^{T}$ is the left Cauchy-Green deformation tensor, $\mathbf{F}$ is the deformation gradient, $\mathbf{g}=\mathbf{G}\left|\mathbf{G}\right|^{-1/3}$, $\left|\mathbf{G}\right|$ is the determinant of $\mathbf{G}$, ρ is the solid density ($\rho=\rho_0\left|\mathbf{G}\right|^{1/2}, \rho_0$ is the reference density), and η is the specific entropy. The energy $e^{h}\left(\rho,\eta\right)$ is the hydrodynamic part of the energy, depending only on the determinant of $\mathbf{G}$ and the entropy η, and $e^{e}\left(\mathbf{g}\right)$ is the shear elastic energy. Eventually, $e^{e}\left(\mathbf{g}\right)$ can depend also on the entropy through the material parameters as the shear modulus, for example. In the present paper, we will consider only isotropic solids, where

$$e^{e}\left(\mathbf{g}\right)=e^{e}\left(j_1,j_2\right),\quad j_1=tr\left(\mathbf{g}\right),\quad j_2=\mathbf{g}:\mathbf{g}=tr\left(\mathbf{g}^2\right).$$

The shear part of the energy is unaffected by the volume change. Such a decomposition into purely volumetric and isochoric deformation is useful, in particular, for description of nearly incompressible isotropic hyperelasticity [8], [13]. In particular, with the energy of the form (1) the pressure is determined only by the hydrodynamic part :

$$p=\rho^2\frac{\partial e^{h}}{\partial\rho},$$

while the deviatoric part $\mathbf{S}$ of the Cauchy stress tensor

$$\sigma=-pI+\mathbf{S},\quad tr\left(\mathbf{S}\right)=0,$$

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is given only by the shear part :

$$\mathbf{S} = -2\rho\left(\frac{\partial e^e}{\partial j_1}\left(\mathbf{g} - \frac{j_1}{3}\mathbf{I}\right) + 2\frac{\partial e^e}{\partial j_2}\left(\mathbf{g}^2 - \frac{j_2}{3}\mathbf{I}\right)\right).$$

An important numerical advantage of giving the energy in separable form is that the numerical non-stationary codes of hyperelasticity can directly be used for fluids (it is sufficient to take $e^e = 0$). Also, such a separable form allows us to prove easier the hyperbolicity of governing equations. The hyperbolicity property is a necessary condition for the wellposedness of the Cauchy problem and the corresponding numerical Godunov's methods. Even if the criterion of hyperbolicity of non-stationary hyperelasticity is well known (the energy e should be an rank-one convex function of the deformation gradient $\mathbf{F}$ [3]), it is extremely difficult, if not impossible, to verify it in practice even in the case of isotropic elastic materials (cf. [2], [4], [5], [9] for the rank-one convexity study). Recently, in the case of isotropic solids with the equation of state in separable form (1), we have proposed a criterion of hyperbolicity in 3D case which is easier to verify [10].

We study here a one - parameter family of energies. For a particular value of the parameter, it contains compressible neo-hookean solids. By using the above mentioned criterion, we will show that in a large domain of parameter the energy is rank-one convex. The result obtained is not new for the case of compressible neo-hookean solids (the proof of the polyconvexity that implies the rank-one convexity can be found in [8]). Finally, a specific Riemann problem (pure torsion test) will be considered to show a strong dependence of the solution on this parameter.

2 One-parameter family of energies

Let $\mathbf{F}$ be the deformation gradient, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy - Green deformation tensor, $\mathbf{G} = (\mathbf{B})^{-1}$ is the Finger tensor. Using notations $\mathbf{a}, \mathbf{b}, \mathbf{c}$ for the columns of $\mathbf{F}^{-1}$ we have :

$$\mathbf{F}^{-1} = (\mathbf{a}, \mathbf{b}, \mathbf{c}), \quad \mathbf{F}^{-T} = \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} ||\mathbf{a}||^2 & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{b} & ||\mathbf{b}||^2 & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & ||\mathbf{c}||^2. \end{pmatrix}.$$

Denoting the determinant of $\mathbf{G}$ by $|\mathbf{G}|$ we obviously have

$$|\mathbf{G}| = \Delta^2,$$

where Δ is the determinant of $\mathbf{F}^{-1}$:

$$\Delta = \left|\mathbf{F}^{-1}\right| = \mathbf{a}\cdot(\mathbf{b} \wedge \mathbf{c}).$$

We introduce a reduced Finger tensor having a unit determinant :

$$\mathbf{g} = \mathbf{G}/\left|\mathbf{G}\right|^{1/3}.$$

Consider, in particular, the following particular example of a one-parameter family of the equations of state in separable form (1) proposed in [6] :

$$e^e = \frac{\mu}{4\rho_0}\left(a j_2 + \frac{1-2a}{3} j_1^2 + 3(a-1)\right). \tag{2}$$

Here a can be viewed as a new non-linear material parameter. In the limit of small deformations, for any value of the parameter a , the Hooke law is recovered. An explicit expression of the invariants j_k in terms of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is :

$$j_1(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{\Delta^{2/3}},$$

$$j_2(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{\|\mathbf{a}\|^4 + \|\mathbf{b}\|^4 + \|\mathbf{c}\|^4 + 2(\mathbf{b}\cdot\mathbf{c})^2 + 2(\mathbf{a}\cdot\mathbf{b})^2 + 2(\mathbf{a}\cdot\mathbf{c})^2}{\Delta^{4/3}}.$$

The hydrodynamic part of the energy $e^h(\rho, \eta)$ can be taken, as a convex function of (τ, η), $\tau = 1/\rho$. For example, the 'stiffened gas' equation of state can be used :

$$e^h(\rho, \eta) = \frac{p + \gamma p_\infty}{\rho(\gamma - 1)}, \quad p + p_\infty = f(\eta)\rho^\gamma, \quad \frac{df}{d\eta} > 0. \tag{3}$$

In particular, one can take

$$f(\eta) = A \exp\left(\frac{\eta}{c_v}\right), \quad A = const > 0.$$

Here $p_\infty = const$, c_v is the heat capacity at constant volume, and $\gamma > 1$ is the polytropic exponent. The hydrodynamic sound speed as a function of the pressure and the density is given by:

$$c^2 = \frac{\gamma(p + p_\infty)}{\rho}.$$

The energy in separable form with $a = 0.5$ was successfully used in [12] for a numerical study of dynamical fracture and fragmentation of metals in the framework of a model of Maxwell type solids. The corresponding isochoric energy is

$$e^e = \frac{\mu}{8\rho_0}(j_2 - 3). \tag{4}$$

The rank-one convexity of (4) was proved in [10]. Also, as it was proved in [6], for any $0 < a < 0.5$ the energy (2) is rank-one convex. It is interesting to note that the value $a = -1$ corresponds to neo-hookean solids :

$$e^e = \frac{\mu}{4\rho_0}\left(j_1^2 - j_2 - 6\right).$$

Indeed, for neo-hookean solids one takes e^e in the form

$$e^e = \frac{\mu}{2\rho_0}(i_1 - 3),$$

where

$$i_1 = tr\left(\overline{\mathbf{B}}\right), \quad \overline{\mathbf{B}} = \mathbf{B}/|\mathbf{B}|^{1/3}.$$

Due to Cayley - Hamilton theorem,

$$\overline{\mathbf{B}}^3 - i_1\overline{\mathbf{B}}^2 + \frac{i_1^2 - i_2}{2}\overline{\mathbf{B}} - \mathbf{Id}_3 = 0, \quad i_2 = \overline{\mathbf{B}} : \overline{\mathbf{B}} = tr\left(\overline{\mathbf{B}}^2\right).$$

It implies

$$\overline{\mathbf{B}}^{-3} - \frac{i_1^2 - i_2}{2}\overline{\mathbf{B}}^{-2} + i_1\overline{\mathbf{B}}^{-1} - \mathbf{Id}_3 = 0.$$

Or, since $\mathbf{g} = \overline{\mathbf{B}}^{-1}$, one has

$$\mathbf{g}^3 - \frac{i_1^2 - i_2}{2}\mathbf{g}^2 + i_1\mathbf{g} - \mathbf{Id}_3 = 0. \tag{5}$$

Since at the same time

$$\mathbf{g}^3 - j_1\mathbf{g}^2 + \frac{j_1^2 - j_2}{2}\mathbf{g} - \mathbf{Id}_3 = 0, \quad j_2 = \mathbf{g} : \mathbf{g} = tr\left(\mathbf{g}^2\right), \tag{6}$$

one can identify the expressions for the invariants to obtain the relation between j_k and i_k, $k = 1,\ 2$:

$$j_1 = \frac{i_1^2 - i_2}{2}, \quad \frac{j_1^2 - j_2}{2} = i_1.$$

Hence, the shear energy of the neo-Hookean materials expressed in terms of invariants j_k is :

$$e^e = \frac{\mu}{2\rho_0}(i_1 - 3) = \frac{\mu}{4\rho_0}\left(j_1^2 - j_2 - 6\right). \tag{7}$$

The rank-one convexity of compressible neo-hookean materials was established, for example, in [8]. Since the neo-hookean materials correspond to the value $a = -1$, the question arises whether the criterion of hyperbolicity proposed in [10] can also be applied to the neo-hookean materials. Also, it would be useful to have, for experimental and numerical purposes, a one-parameter family (2) that is rank - one convex for a larger interval of a, for example for all a from $[-1, 0.5]$.

3 Rank-one convexity

Theorem 4 in [10] gives a criterium of hyperbolicity in 3D case. In particular, in the isentropic case it states that if the matrix $\mathbf{M}$ given by:

$$\mathbf{M} = \frac{1}{\Delta}\mathbf{F}^{-T}E''\mathbf{F}^{-1} = \frac{1}{\Delta}\begin{pmatrix} \mathbf{a}^T E''\mathbf{a} & \mathbf{a}^T E''\mathbf{b} & \mathbf{a}^T E''\mathbf{c} \\ \mathbf{a}^T E''\mathbf{b} & \mathbf{b}^T E''\mathbf{b} & \mathbf{b}^T E''\mathbf{c} \\ \mathbf{a}^T E''\mathbf{c} & \mathbf{b}^T E''\mathbf{c} & \mathbf{c}^T E''\mathbf{c} \end{pmatrix} \tag{8}$$

is positive definite in the domain

$$X^2 + Y^2 + Z^2 - 2XYZ < 1, \tag{9}$$

and the squared hydrodynamic sound velocity $c^2 = \dfrac{\partial p}{\partial \rho}$ is positive (it is equivalent that e^h is convex with respect to $\tau = 1/\rho$), then the the equations are hyperbolic, i.e. the energy $e = e^h + e^e$ is rank-one convex. Here X, Y and Z are cosinus of angles between vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$:

$$X = \cos(\mathbf{a}, \mathbf{b}), \quad Y = \cos(\mathbf{a}, \mathbf{c}), \quad Z = \cos(\mathbf{b}, \mathbf{c}),$$

E'' is the Hessian matrix of the volume isochoric energy $E = \Delta e^e$ with respect to $\mathbf{a}$ (due to the invariance of E with respect to rotation, one can take also the Hessian matrix with respect to $\mathbf{b}$ or $\mathbf{c}$). The domain (9) corresponds to all possible deformations with

$$\det \mathbf{F} = 1, \tag{10}$$

i.e. corresponding to incompressible solids. Indeed, due to the fact that the shear specific energy e^e is a homogeneous function of degree zero with respect to $\mathbf{G}$, the matrix $\mathbf{M}$ has the same property. So, the restriction (10) is natural.

Since $\mu/(4\rho_0) > 0$, we take it one in (7). Then, up to a linear function of $\mathbf{a}$,

$$E = \Delta\left(j_1^2 - j_2\right) = 2\frac{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 + \|\mathbf{a}\|^2\|\mathbf{c}\|^2 - (\mathbf{a}\cdot\mathbf{c})^2 + \|\mathbf{b}\|^2\|\mathbf{c}\|^2 - (\mathbf{b}\cdot\mathbf{c})^2}{\Delta^{1/3}}.$$

The first derivative of E is :

$$\frac{\partial E}{\partial \mathbf{a}} = 4\frac{\|\mathbf{b}\|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b} + \|\mathbf{c}\|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\,\mathbf{c}}{\Delta^{1/3}}$$

$$-\frac{2}{3}\left(\frac{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 + \|\mathbf{a}\|^2\|\mathbf{c}\|^2 - (\mathbf{a}\cdot\mathbf{c})^2 + \|\mathbf{b}\|^2\|\mathbf{c}\|^2 - (\mathbf{b}\cdot\mathbf{c})^2}{\Delta^{4/3}}\right)(\mathbf{b}\wedge\mathbf{c}).$$

The second derivative of E is :

$$E'' = \frac{\partial^2 E}{\partial \mathbf{a}^2} = 4\frac{\left(\|\mathbf{b}\|^2 + \|\mathbf{c}\|^2\right)\mathbf{I} - \mathbf{b}\otimes\mathbf{b} - \mathbf{c}\otimes\mathbf{c}}{\Delta^{1/3}}$$

$$-\frac{4}{3}\left(\frac{\|\mathbf{b}\|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b} + \|\mathbf{c}\|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\,\mathbf{c}}{\Delta^{4/3}}\right)\otimes(\mathbf{b}\wedge\mathbf{c}) - \frac{4}{3}(\mathbf{b}\wedge\mathbf{c})\otimes\left(\frac{\|\mathbf{b}\|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b} + \|\mathbf{c}\|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\,\mathbf{c}}{\Delta^{4/3}}\right)$$

$$+\frac{8}{9}\left(\frac{\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2+\|\mathbf{a}\|^2\|\mathbf{c}\|^2-(\mathbf{a}\cdot\mathbf{c})^2+\|\mathbf{b}\|^2\|\mathbf{c}\|^2-(\mathbf{b}\cdot\mathbf{c})^2}{\Delta^{7/3}}\right)(\mathbf{b}\wedge\mathbf{c})\otimes(\mathbf{b}\wedge\mathbf{c})\,.$$

It implies :

$$\mathbf{a}^T E''\mathbf{a}=\frac{\frac{20}{9}\left(\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2\right)+\frac{20}{9}\left(\|\mathbf{a}\|^2\|\mathbf{c}\|^2-(\mathbf{a}\cdot\mathbf{c})^2\right)+\frac{8}{9}\left(\|\mathbf{b}\|^2\|\mathbf{c}\|^2-(\mathbf{b}\cdot\mathbf{c})^2\right)}{\Delta^{1/3}},$$

$$\mathbf{a}^T E''\mathbf{b}=\frac{8}{3}\frac{\left((\mathbf{a}\cdot\mathbf{b})\|\mathbf{c}\|^2-(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{c})\right)}{\Delta^{1/3}},$$

$$\mathbf{a}^T E''\mathbf{c}=\frac{8}{3}\frac{\left((\mathbf{a}\cdot\mathbf{c})\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})\right)}{\Delta^{1/3}},$$

$$\mathbf{b}^T E''\mathbf{c}=0,$$

$$\mathbf{b}^T E''\mathbf{b}=4\frac{\left(\|\mathbf{b}\|^2\|\mathbf{c}\|^2-(\mathbf{b}\cdot\mathbf{c})^2\right)}{\Delta^{1/3}},$$

$$\mathbf{c}^T E''\mathbf{c}=4\frac{\left(\|\mathbf{b}\|^2\|\mathbf{c}\|^2-(\mathbf{b}\cdot\mathbf{c})^2\right)}{\Delta^{1/3}}.$$

Finally, the matrix $\mathbf{M}$ at the surface $\Delta=1$ can be written as :

$$\mathbf{M}=\mathbf{D}\mathbf{N}\mathbf{D}^T$$

where

$$\mathbf{D}=\mathbf{D}^T=\begin{pmatrix}\frac{2}{3}&0&0\\0&2&0\\0&0&2\end{pmatrix},$$

$$\mathbf{N}=\begin{pmatrix}5A^2B^2\left(1-X^2\right)+5A^2C^2\left(1-Y^2\right)+2B^2C^2\left(1-Z^2\right)&2ABC^2\left(X-YZ\right)&2AB^2C\left(Y-XZ\right)\\2ABC^2\left(X-YZ\right)&B^2C^2\left(1-Z^2\right)&0\\2AB^2C\left(Y-XZ\right)&0&B^2C^2\left(1-Z^2\right)\end{pmatrix}.$$

Here we denoted

$$A=\|\mathbf{a}\|\,,\quad B=\|\mathbf{b}\|\,,\quad C=\|\mathbf{c}\|\,.$$

The positive definiteness of $\mathbf{M}$ is equivalent to the positive definiteness of the matrix $\mathbf{N}$.

$$A=\|\mathbf{a}\|\,,\qquad B=\|\mathbf{b}\|\,,\qquad C=\|\mathbf{c}\|\,.$$

Estimating $\mathbf{N}$ in the Cartesian basis (corresponding to $X=Y=Z=1$ and $A=B=C=1$) we obtain

$$\mathbf{N}=\begin{pmatrix}12&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Hence, $\mathbf{N}$ is positive definite at this point. It is now sufficient to show that the determinant of $\mathbf{N}$ is positive to assure that $\mathbf{N}$ is positive definite. The determinant of $\mathbf{N}$ is :

$$\det\mathbf{N}=2B^6C^6\left(1-Z\right)^3\left(1+Z\right)^3$$

$$+A^2B^4C^6\left(1-Z\right)\left(1+Z\right)\left(Y^2Z^2-5Z^2+8XYZ-5Y^2-4X^2+5\right)$$

$$+A^2B^6C^4(1-Z)(1+Z)\left(X^2Z^2-5Z^2+8XYZ-4Y^2-5X^2+5\right)$$

$$=B^4C^4\left(1-Z^2\right)\left(2B^2C^2\left(1-Z^2\right)^2+A^2C^2\left(Y^2Z^2-5Z^2+8XYZ-5Y^2-4X^2+5\right)\right.$$

$$+ A^2B^2(X^2Z^2 - 5Z^2 + 8XYZ - 4Y^2 - 5X^2 + 5)\big)$$

Obviously, in the domain of (X, Y, Z) defined by (9) the determinant is positive because

$$\begin{aligned} &Y^2Z^2 - 5Z^2 + 8XYZ - 5Y^2 - 4X^2 + 5 \\ &= Y^2Z^2 - Z^2 - Y^2 + 1 - 4\left(X^2 + Y^2 + Z^2 - 2XYZ - 1\right) \\ &> \left(1 - Y^2\right)\left(1 - Z^2\right) > 0, \\ &X^2Z^2 - 5Z^2 + 8XYZ - 4Y^2 - 5X^2 + 5 \\ &= X^2Z^2 - Z^2 - X^2 + 1 - 4\left(X^2 + Y^2 + Z^2 - 2XYZ - 1\right) \\ &> \left(1 - X^2\right)\left(1 - Z^2\right) > 0. \end{aligned}$$

This proves the hyperbolicity of the generalized neo-hookean materials. Even if the proof is a little bit lengthy for this simple case of neo-hookean materials, it follows the same line as the case $a = 0.5$ for which exiting in the litterature sufficient criteria of rank-one convexity fail.

The final remark is that the energy equation (2) can also be written as a linear combination of j_2 and the energy of neo-hookean solids:

$$e^e = \frac{\mu}{4\rho_0}\left(\frac{1+a}{3}(j_2 - 3) + \frac{1-2a}{3}\left(j_1^2 - j_2 - 6\right)\right).$$

Since the matrices M corresponding to the energies j_2 and the neo-hookean solids (7) are non negatives, the same property will be valid for the one-parameter family of energies (2) for any a : $-1 \le a < 0.5$.

4 Influence of the parameter a

For applications to rubbers one can take, for example, the following values of the material parameters corresponding to silastic RTV-521 :

$$\rho_0 = 1372 \quad kg/m^3, \qquad \mu = 1 \quad MPa, \quad \gamma = 2.4, \quad p_\infty = 3.3 \quad GPa.$$

The experimental data for determining the hydrodynamic part of the energy are taken from the database [1].

The behavior of the dimensionless deviatoric stress S_{11}/μ as a function of the strain $1-a_1$ (a_1 is the first component of the vector $\mathbf{a} = (a_1, a_2, a_3)^T$) is shown in Figure 1. A one dimensional shear test case is addressed below. The studied configuration is shown in Figure 2 where an elastic body is subjected to a strong shear test. The same computation is performed with different values of parameter a. The hyperelastic non-stationary model was used for numerical solving the Riemann problem. The considered mesh involves 4000 cells. The Riemann problem has been solved using a robust splitting method described in [6]. The exact solution can also be constructed in this case [11]. One can see on Figures 2 and 3 that the velocities of shear waves (shocks) are smaller for neo-hookean solids. Also, the initial shear discontinuity produces much stronger normal velocity jump when $a = 0.5$ compared to the neo-hookean solids. The shear stress amplitude is smaller in the neo-hookean solids, in spite of large transverse deformations.

5 Conclusions

Applying the theorem proved in [10], we established the hyperbolicity of dynamic equations of hyperelastic isotropic solids in the case of a one-parameter family of equations of state (2) containing, in paricular, generalized neo-hookean solids. The hyperbolicity is equivalent to the rank-one convexity of the corresponding energies. The influence of the parameter on the solution properties is shown in the case of a strong shear test.

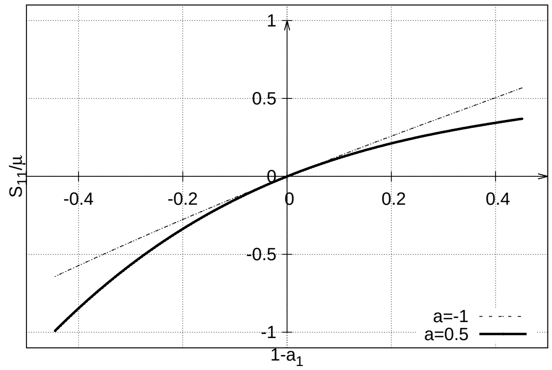

Figure 1: The behaviour of the dimensionless deviatoric stress S_{11}/μ is shown. Solid line corresponds to $a = 0.5$, while the dotted line (which is almost a straight line) corresponds to the neo-hookean solids ($a = -1$).

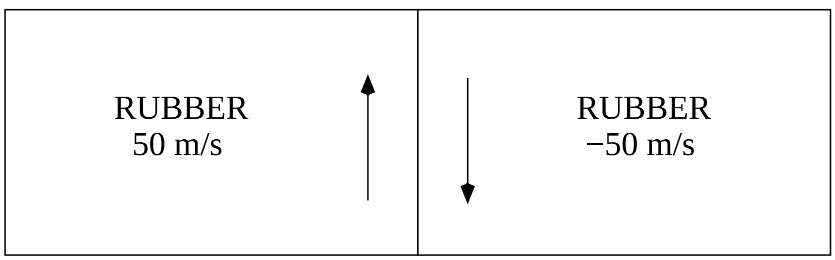

Figure 2: An elastic solid (rubber) is subjected to a shear test. To the left, solids admits a positive tangential velocity while to the right, its velocity is negative.

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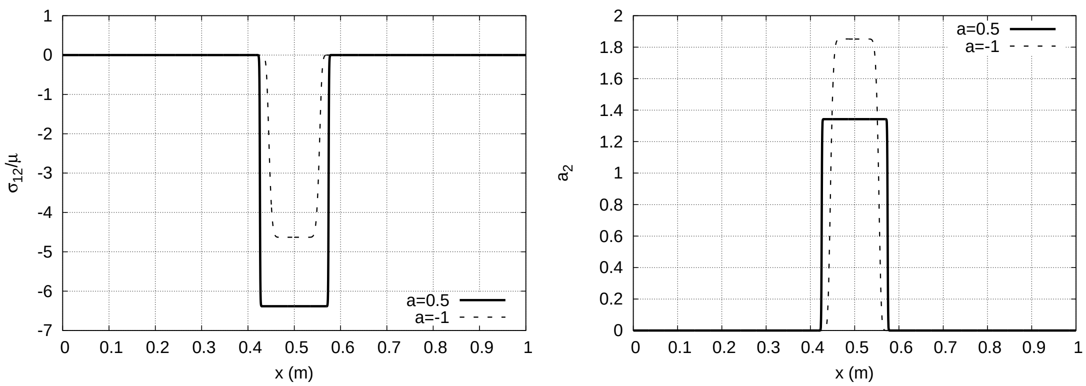

Figure 3: Shear stress normalized by the shear modulus is plotted on the left for two values of the parameter a at the same time ($t = 2\ ms$). On the right, the component a_2 of the vector $\mathbf{a}$ is shown. Solid line corresponds to $a = 0.5$, while the dotted line corresponds to the neo-hookean solids ($a = -1$).

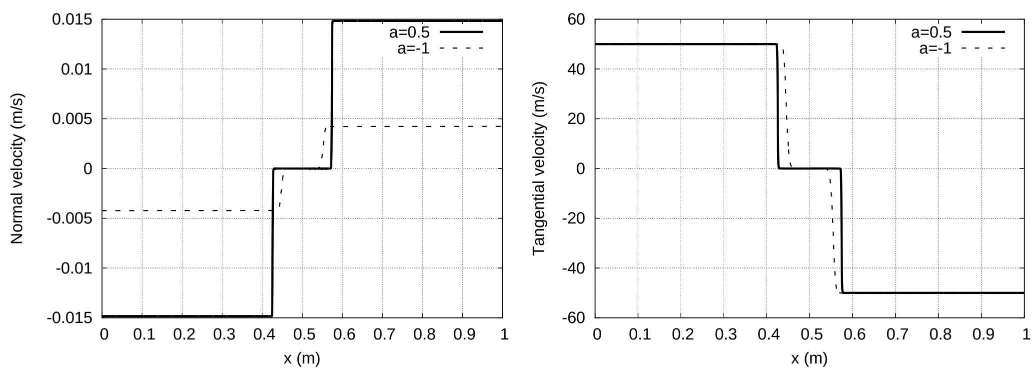

Figure 4: The normal velocity is plotted on the left, the tangential velocity is plotted on the right. Solid line corresponds to $a = 0.5$, while the dotted line corresponds to the neo-hookean solids ($a = -1$).

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